ELSEVIER

Journal of Pure and Applied Algebra 114 (1997) 175–208

JOURNAL OF PURE AND APPLIED ALGEBRA

The groups $(l,m \mid n,k)$

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Communicated by P.T. Johnstone; received 23 June 1995; revised 24 October 1995

Abstract

The groups $(l,m \mid n,k)$ defined by the presentations

$$\langle a,b : a^l = b^m = (ab)^n = (ab^{-1})^k = 1\rangle,$$

were first studied systematically by Coxeter in 1939, and have been a subject of interest ever since, particularly with regard to the question as to which of them are finite. The finiteness question has been completely determined for $l = 2$ and $l = 3$, and there are some other partial results. In this paper, we give a complete determination as to which of the groups $(l,m \mid n,k)$ are finite.

The proof of this result essentially splits into two parts. When l, m, n and k are "large" (in a sense to be made precise in the paper), we can use arguments in terms of pictures to show that $(l,m \mid n,k)$ is infinite; this will involve finding generators for the second homotopy modules of the presentations. For small values of l, m, n and k, the groups are finite, and we can quote previously established results. For intermediate values, the groups can still be infinite, even though the arguments in terms of pictures do not apply. In these cases, where the status of the group was previously open, we produce a series of individual arguments to show that the groups are infinite; many of these are based on computational results.

1991 Math. Subj. Class.: 20F05

1. Introduction

The group $(l,m \mid n,k)$ is defined by the presentation

$$\langle a,b : a^l = b^m = (ab)^n = (ab^{-1})^k = 1\rangle,$$

where $l,m,n,k > 1$. It is clear that $(l,m \mid n,k)$ is isomorphic to $(m,l \mid n,k)$ and also to $(l,m \mid k,n)$, and so we may assume, without loss of generality, that $l \leq m$ and that $n \leq k$; we will adopt this convention throughout this paper.

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SSDI 0022-4049(95)00164-6

This class of groups were first studied systematically in [3], where several results on their finiteness (or otherwise) were proved. As we shall note in a moment, the finiteness question has been completely decided for $l = 2$ and $l = 3$, and there are some other partial results. The purpose of this paper is to build on these results and give a complete classification as to which of the groups $(l,m \mid n,k)$ are finite.

We should mention in passing that the question of the finiteness of the group $(m,n,p;q)$ defined by the presentation

$$\langle a,b : a^m = b^n = (ab)^p = (a^{-1}b^{-1}ab)^q = 1 \rangle$$

has been largely determined, in that there are (at present) only six values of $(m,n,p;q)$ for which the finiteness or otherwise of the group is an open question; see [13, 16, 6–8] (and also [17, 24] for a survey of such results). We will make use of part of this classification in this paper (see Theorems 2.7 and 2.8).

Returning to $(l,m \mid n,k)$, the finiteness question has been settled (as we mentioned above) for $l = 2$, in that we have:

Proposition 1.1. *The group $(2,m \mid n,k)$ is isomorphic to the triangle group $(2,m,d)$, where $d = \gcd(n,k)$, and is thus finite if and only if $1/m + 1/d > \frac{1}{2}$.*

This result was noted in [3]. As far as the case $l = 3$ is concerned, $(3,m \mid n,k)$ is isomorphic to $(3,n \mid m,k)$; so, given Proposition 1.1, we may assume that $m \geq 3$ and that $k \geq n \geq 3$, and we have the following result from [4]:

Proposition 1.2. *If $m \geq 3$ and $k \geq n \geq 3$, then $(3,m \mid n,k)$ is finite if and only if*

$$\cos\left(\frac{2\pi}{m}\right) + \cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{2\pi}{k}\right) < 0.$$

Given these two results, we only need consider the cases where $m \geq l \geq 4$. In this paper, we will prove the following:

Theorem 1.3. *If $m \geq l \geq 4$ and $k \geq n \geq 2$, then $(l,m \mid n,k)$ is finite if and only if one of the following possibilities occurs:*

1. $n = k = 2$;
2. $l = m = 4$, $n = 2$;
3. (l,m,n,k) *is one of*
   $(4,5,2,k), 3 \leq k \leq 5$; $(4,m,2,3), 6 \leq m \leq 9$; $(5,m,2,3), m \geq 5$;
   $(5,5,2,4)$; $(6,7,2,3)$; $(7,7,2,3)$;
   $(7,8,2,3)$.

Combining Theorem 1.3 with Propositions 1.1 and 1.2 yields:

Theorem 1.4. *If* $m \geq l \geq 2$, $k \geq n \geq 2$, $d = \gcd(n,k)$ *and* $e = \gcd(m,k)$, *then* $(l,m \mid n,k)$ *is finite if and only if* (l,m,n,k) *is one of*:

$(2,2,n,k), n \geq 2$;	$(2,m,n,k), m \geq 3, d \leq 2$;
$(2,3,n,k), 3 \leq d \leq 5$;	$(2,m,n,k), 4 \leq m \leq 5, d = 3$;
$(3,m,2,k), e \leq 5$;	$(3,3,3,k), k \geq 3$;
$(3,3,4,4)$;	$(3,3,4,5)$;
$(3,4,3,4)$;	$(3,4,3,5)$;
$(3,5,3,4)$;	$(3,m,3,3), m \geq 4$;
$(l,m,2,2), l \geq 4$;	$(4,4,2,k), k \geq 3$;
$(4,5,2,k), 3 \leq k \leq 5$;	$(4,m,2,3), 6 \leq m \leq 9$;
$(5,m,2,3), m \geq 5$;	$(5,5,2,4)$;
$(6,7,2,3)$;	$(7,7,2,3)$;
$(7,8,2,3)$.	

The proof of Theorem 1.3 essentially splits into two parts. When l, m, n and k are "large" (in a sense to be made precise later), we can use arguments in terms of pictures to show that $(l,m \mid n,k)$ is infinite; this will involve finding generators for the second homotopy modules of the presentations. We give an account of pictures in Section 3, and apply these to our groups in Sections 4 and 5. For small values of l, m, n and k, the groups are finite, and we can quote previously established results. For intermediate values, the groups can still be infinite, even though the arguments in terms of pictures do not apply. In these cases, where the status of the group was previously open, we produce a series of individual arguments to show that the groups are infinite. Many of these are based on computational results; the main software we used was Cayley [2], GAP [21] and Quotpic [14], and software provided by Edmund Robertson, which included a Todd–Coxeter program, a Reidemeister–Schreier routine based on [10] and the Tietze transformation program described in [11]. A nice overview of the use of these programs may be found in [12]. We would like to acknowledge the role played by all these programs in the proof of Theorem 1.3.

2. Preliminary results

In this section, we shall list some previous results about these groups which we will combine with the results proved in this paper to produce Theorem 1.3. We also list some general results for proving groups infinite.

We start with two results which state that certain of these groups are finite. The first may be found in [3]:

Proposition 2.1. *The following groups are all finite*:

1. $(4,4 \mid 2,k)$, *which has order* $4k^2$ *for all* k;
2. $(4,5 \mid 2,3)$, *which is trivial*;
3. $(4,5 \mid 2,4)$, *which is a semi-direct product of* E_{32} *by* C_5, *and hence has order* 160;

4. $(4,5\,|\,2,5)$, *which is isomorphic to* A_6;
5. $(4,6\,|\,2,3)$, *which is isomorphic to* S_5;
6. $(4,7\,|\,2,3)$, *which is isomorphic to* $PSL(2,7)$;
7. $(4,8\,|\,2,3)$, *which has order* 1152; *this group* G *has abelian quotient* C_4 *and* G' *is the central product of two copies of* $SL(2,3)$;
8. $(4,9\,|\,2,3)$, *which is isomorphic to* $PSL(2,17)$;
9. $(5,m\,|\,2,3)$, *which is isomorphic to* A_5 *if* 5 *divides* m, *and is trivial otherwise*;
10. $(5,5\,|\,2,4)$, *which is isomorphic to* A_6;
11. $(l,m\,|\,2,2)$, *which is trivial if* l *and* m *are odd, dihedral of order* $2m$ *if* l *is even and* m *odd, dihedral of order* $2l$ *if* m *is even and* l *odd, and metabelian of order* $4lm$ *if* l *and* m *are even*;
12. $(6,7\,|\,2,3)$, *which is isomorphic to* $PSL(2,13)$;
13. $(7,7\,|\,2,3)$, *which is isomorphic to* $PSL(2,13)$.

Here C_m denotes the cyclic group of order m and E_n the elementary abelian group of order n; A_n denotes the alternating group of degree n and S_n the symmetric group of degree n. From [18], we have the following:

Theorem 2.2. *The group* $(7,8\,|\,2,3)$ *is a semi-direct product of* E_{64} *by* $PSL(2,7)$, *and hence is finite of order* 10752.

The following two results were proved in [3]:

Proposition 2.3. *The group* $(5,m\,|\,2,k)$ *is isomorphic to* $(5,k\,|\,2,m)$.

Theorem 2.4. *If* l *and* m *are even, or if* l *and* k *are even with* $n=2$, *then* $(l,m\,|\,n,k)$ *is finite if and only if*

$$2\sin\left(\frac{\pi}{l}\right)\sin\left(\frac{\pi}{m}\right) > \cos\left(\frac{\pi}{n}\right) + \cos\left(\frac{\pi}{k}\right).$$

Since $(l,m\,|\,n,k)$ is isomorphic to $(m,l\,|\,n,k)$, we also have the conclusion of Theorem 2.4 holding when m and k are even and $n=2$. This result clearly reduces the problem considerably, and we spell out the consequences of it explicitly.

Corollary 2.5. *If* $4 \le l \le m$ *and* $2 \le n \le k$, *and if one of the following conditions holds*: (1) l *and* m *are even*; (2) l *and* k *are even with* $n=2$; (3) m *and* k *are even with* $n=2$, *then* $(l,m\,|\,n,k)$ *is finite if and only if either* $l=m=4$ *and* $n=2$, *or else* (l,m,n,k) *is one of* $(4,5,2,4)$, $(4,6,2,3)$ *and* $(4,8,2,3)$.

The following result is essentially in [13]:

Theorem 2.6. $(4,5\,|\,2,k)$ *is finite if and only if* $k \le 5$.

Proof. In [13], it is shown that the group with presentation

$$\langle x, y : x^2 = y^4 = (xy)^5 = (xy^2)^k = 1\rangle$$

is finite if and only if $k \leq 5$. Writing the presentation in terms of new generators $a = y$ and $b = xy$ yields the result. □

We also have the following:

Theorem 2.7. *$(7, m \mid 2, 3)$ is infinite for $m \geq 9$.*

Proof. In [3], it is shown that the group $G^{k,l,2m}$ with presentation

$$\langle r, s, t : r^k = s^l = t^{2m} = (rs)^2 = (st)^2 = (tr)^2 = (rst)^2 = 1\rangle$$

contains the group $H(k, l, m)$ with presentation

$$\langle x, y : x^k = y^l = (xy)^2 = (x^2y^2)^m = 1\rangle$$

as a subgroup of index 2. In the case where $k = 3$ and $l = 7$, we have the group $G^{3,7,2m}$, which contains $(2, 3, 7; m)$ as a subgroup of index 2, and which is therefore infinite for $m \geq 9$ by [13, 16, 6]. On the other hand, if we put $u = y$ and $v = xy^{-2}$, then the presentation for $H(3, 7, m)$ becomes

$$\langle u, v : (vu^2)^3 = u^7 = (vu^3)^2 = v^m = 1\rangle;$$

introducing $w = u^3$ and deleting u transforms this to

$$\langle w, v : (vw^3)^3 = w^7 = (vw)^2 = v^m = 1\rangle.$$

Given $(vw)^2 = 1$, the relation $(vw^3)^3 = 1$ is equivalent to $(w^{-1}v^{-1}w^2)^3 = 1$, and hence to $(v^{-1}w)^3 = 1$. So we have that $H(3, 7, m)$ is isomorphic to $(7, m \mid 2, 3)$, and the result follows. □

We now prove a result classifying the groups $(l, m \mid n, k)$ with $l = m$. If $l = 2$ or $l = 3$, the result follows from Propositions 1.1 and 1.2, and so we concentrate on the case where $l \geq 4$.

Theorem 2.8. *The group $(l, l \mid n, k)$ with $l \geq 4$, $n \leq k$ and $(n, k) \neq (2, 2)$ is finite if and only if one of the following four possibilities occurs: $l = 4, n = 2, k \geq 3$; $l = 5, n = 2, k = 3$; $l = 5, n = 2, k = 4$; $l = 7, n = 2, k = 3$.*

Proof. The group $G = (l, l \mid n, k)$ has presentation

$$\langle a, b : a^l = b^l = (ab)^n = (ab^{-1})^k = 1\rangle.$$

As was pointed out in [3], this group has an automorphism t of order 2 interchanging a and b. If we form the semi-direct product of G with $\langle t\rangle$, we get the group with presentation

$$\langle t, a : t^2 = a^l = (at)^{2n} = (t^{-1}a^{-1}ta)^k = 1\rangle,$$

i.e. the group $(2, l, 2n; k) = (2, l, 2k; n)$. By [7], and since $n \le k$, this group is finite if and only if we have one of the four possibilities listed above. □

Following [3], we say that the group $(l, m \mid n, k)$ *collapses* if any of the elements a, b, ab and ab^{-1} have orders less than l, m, n and k respectively, and otherwise that the group *does not collapse*. From [23], we immediately deduce the following result:

Proposition 2.9. *If $(l, m \mid n, k)$ does not collapse, and if $1/l + 1/m + 1/n + 1/k \le 1$, then $(l, m \mid n, k)$ is infinite.*

Lastly, we have the following variation on the Golod–Šafarevič theorem from [25]:

Theorem 2.10. *If a group G has a finite presentation with n generators and r relators, if G has an elementary abelian p-quotient of rank d for some prime p, and if $r - n \le d^2/4 - d$, then G is infinite.*

3. Pictures

A vital ingredient in the proof of Theorem 1.3 is that of *pictures*, and, for the benefit of the reader, we will give a brief account of these here. For reasons of space, we will not attempt to give a comprehensive treatment of this topic, and refer the reader to [9, 15, 16, 19, 20].

We are considering the group $G = (l, m \mid n, k)$ defined by the presentation

$$\wp = \langle a, b : a^l = b^m = (ab)^n = (ab^{-1})^k = 1\rangle,$$

where $m \ge l \ge 4$ and $k \ge n \ge 2$. Let A and B be the cyclic groups C_l and C_m with presentations $\langle a : a^l = 1\rangle$ and $\langle b : b^m = 1\rangle$ respectively. We can regard G as a *two-relator product* of A and B, that is to say the quotient of the free product $A * B$ by the normal closure of the two words $\alpha = (ab)^n$ and $\beta = (ab^{-1})^k$. We will work with pictures over this two-relator product as in [16]; this is a direct analogy of pictures over one-relator products described in detail in [15]; one should also compare the use of pictures over "relative presentations" in [1]. The technique is originally due to Short [22].

A picture Π over $\wp$ consists of the following:

1. a disc D^2 (with boundary ∂D^2);
2. a collection V of pairwise disjoint closed discs in the interior of D^2 called *vertices*;
3. a finite collection E of pairwise disjoint arcs in the interior of D^2 called *edges*; each edge is either a simple closed curve in the interior of D^2 which meets no

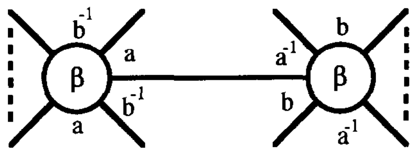


Fig. 1.

vertex, an arc joining two (not necessarily distinct) vertices, an arc joining a vertex to ∂D^2, or an arc joining ∂D^2 to ∂D^2;

4. a collection of *labels*, one at each corner of each *region* of Π (i.e. each connected component of $\partial D^2 - (V \cup E)$), and one along each component of the intersection of a region with ∂D^2.

Each label of Π is one of $\{a, a^{-1}, b, b^{-1}\}$, with the possible exception of labels on (segments of) ∂D^2, which may be any element of $A \cup B$. Reading the labels round a vertex in a clockwise version yields (up to cyclic permutation) α, α^{-1}, β or β^{-1} as a cyclically reduced word in $A * B$; we use the terms α-vertex and β-vertex to denote vertices with label $\alpha^{\pm 1}$ and $\beta^{\pm 1}$ respectively.

A region of Π is called a *boundary* region if it meets ∂D^2, and is said to be *interior* otherwise. If ∂D^2 meets no edges, then Π is said to be *spherical*, and, in this case, ∂D^2 is one of the boundary components of a non-simply connected region called the *distinguished* region (providing, of course, that Π contains at least one vertex); all other regions are interior in this case.

The labels of any region Δ of Π must either all belong to A or all belong to B; thus we have *A-regions* and *B-regions*. The product of all the labels in Δ, evaluated in A or B as appropriate, must be the identity element. Since the labels round each vertex spell out a cyclically reduced word in which the labels come from alternately A and B, each edge must separate an A-region from a B-region.

The boundary label of Π is the cyclically reduced word obtained by reading the labels on ∂D^2 in an *anticlockwise* direction; this word then represents the identity element of G. If Π is spherical, then the boundary label is an element of $A \cup B$ determined by the other labels of the distinguished region.

Two distinct vertices of Π are said to *cancel* along an edge e if they are joined by e and if their labels, read from the endpoints of e, are mutually inverse words in $A * B$; an example of two cancelling β-vertices is given in Fig. 1.

Cancelling vertices can be removed from a picture by a sequence of so-called *bridge* moves and the deletion of a *dipole* (a connected spherical picture over $\wp$ containing exactly two vertices) without changing the boundary label; see [16] for details. Such a cancellation yields an alternative picture with the same boundary label and two fewer vertices.

A picture is said to be *reduced* if it cannot be altered by bridge moves to a picture with a pair of cancelling vertices. Any cyclically reduced word in $A * B$ representing the identity element of G occurs as the boundary label of some such picture. A picture which can be so altered fails to be reduced.

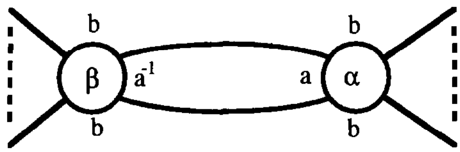


Fig. 2.

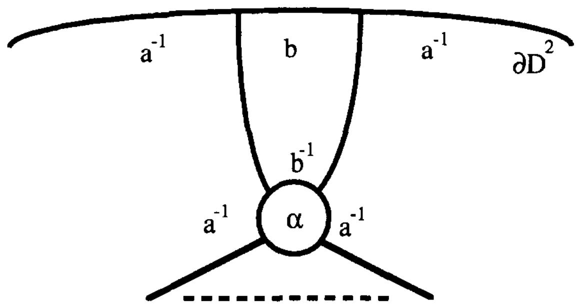


Fig. 3.

The picture Π is said to be *connected* if $V \cup E$ is connected; in particular, no edge of a connected picture is a simple closed curve or connects two points on ∂D^2 unless the picture consists only of that edge.

We now consider parallel edges. Two edges of a picture Π are said to be *parallel* if they form the boundary of a two-sided region, and are called *∂-parallel* if, in addition, they meet ∂D^2. (In this last case, either the region is bounded by two edges and a segment of ∂D^2 or else both edges are arcs joining ∂D^2 to ∂D^2.) If we have two parallel edges joining α-vertices u and v, then u and v will cancel; so, in a minimal situation, we will not have parallel edges joining two α-vertices, and, similarly, we will not have parallel edges joining two β-vertices.

The maximum number of parallel edges between an α-vertex and a β-vertex is two (see Fig. 2), and, in the special case we consider in the proof of Theorem 5.1, the maximum number of ∂-parallel edges is also two (see Fig. 3). The first of these assertions is obvious. For the latter, the picture we consider has boundary label $(a^{-1}b)^u$ where $0 < u < k$, and so the only way to obtain more than two ∂-parallel edges would be to join ∂D^2 to a β-vertex, v say. Suppose that this were the case. It can be assumed, using bridge moves if necessary, that there are $2n-1$ ∂-parallel edges incident at v. Now simply delete v and the $2n-1$ ∂-parallel edges and form a new picture with boundary label $(a^{-1}b)^{k-u}$ but with one fewer β-vertex. This would contradict our minimality condition.

By making moves of the kind shown in Fig. 4 in which γ denotes α or β, we can make the following assumption, which will hold throughout what follows: the label of any region of Π of degree at least three does not contain a substring bb^{-1} or $b^{-1}b$.

We now form a graph $\hat{\Gamma}$ from Π by identifying each pair of parallel or ∂-parallel edges; in this way, we obtain a tesselation of the disc D^2. The vertices of $\hat{\Gamma}$ are simply

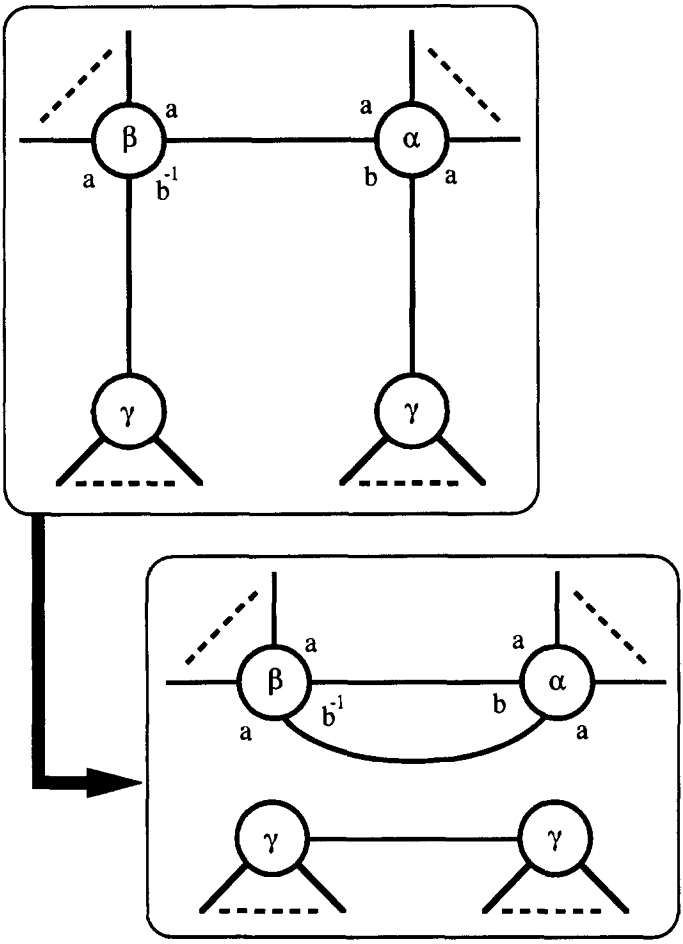


Fig. 4.

the α-vertices and β-vertices of Π, together with vertices on ∂D^2 which we will call *boundary* vertices. A region in $\hat{\Gamma}$ is interior if it was interior in Π; otherwise it is a boundary region. In the case $n = 2$, we make an additional transformation at each α-vertex of Π of degree two of the type shown in Fig. 5.

If Π is a spherical picture, we form a graph Γ from $\hat{\Gamma}$ by contracting the boundary ∂D^2 to a point and removing it; thus Γ will form a tesselation of the sphere S^2 in which each region is a topological disc. The distinguished region of Γ is that obtained from the distinguished region of Π, and all other regions of Γ are said to be interior. If Π is non-spherical, we simply take Γ to be $\hat{\Gamma}$.

Since the maximum number of parallel edges between any two vertices of Π or between a vertex and the boundary was two, it follows that every α-vertex in Γ has degree at least n and that every β-vertex in Γ has degree at least k. Moreover, if any α-vertex u has degree n, then all the vertices adjacent to u in Γ must be β-vertices

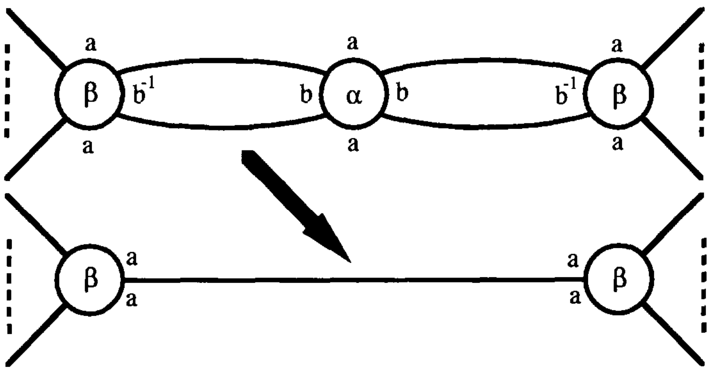


Fig. 5.

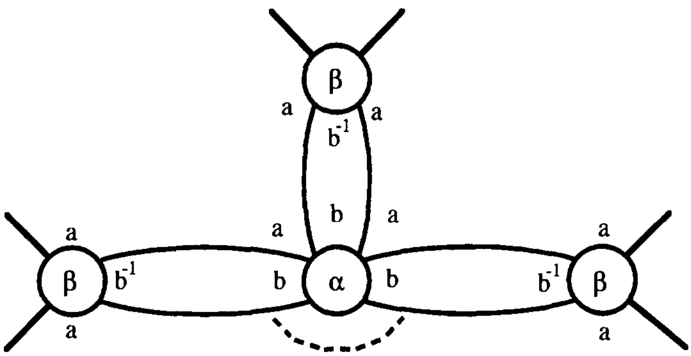


Fig. 6.

(see Fig. 6), and, if any β-vertex v has degree k, then all the vertices adjacent to v in Γ must be α-vertices. However, as we will see later, there will be some instances where we will make a further identification of edges in Γ, and, in this case, we may have β-vertices of degree less than k.

A region Δ of Γ is called an $\tilde{A}$-region if the label of Δ is $a^{\varepsilon_1}a^{\varepsilon_2}a^{\varepsilon_3}a^{\varepsilon_4}$, where $\varepsilon_i = \pm 1$ $(1 \le i \le 4)$ and $\sum_{i=1}^{4} \varepsilon_i = 0$. We see from Fig. 6 that an $\tilde{A}$-region cannot contain any α-vertices of degree n, or, similarly, any β-vertices of degree k.

We now turn to the topic of *curvature*; our curvature arguments are based on those in [5–7]. We give a brief description here, but we refer the reader to those papers for further details.

If we have a vertex u in Γ of degree d, we assign an *angle* of $2\pi/d$ to the d corners of regions around u. If we have a region Δ of Γ of degree k (i.e. a region surrounded by k vertices) where the vertices have degrees d_i $(1 \le i \le k)$, then the curvature $c(\Delta)$ of Δ is defined to be

$$(2-k)\pi \; + 2\pi \sum_{i=1}^{k} \frac{1}{d_i}.$$

We will sometimes denote this by $c(d_1, ..., d_k)$. It follows from Euler's formula that, if Π is a connected spherical picture, then the sum of the curvature of the regions of Γ is 4π; if Π is non-spherical, then the corresponding sum is at least 2π.

In our proofs, we will usually try and show that the values of 4π and 2π cannot be obtained. Our method will be to locate those interior regions Δ of Γ of positive curvature (i.e. those regions Δ with $c(\Delta) > 0$), and show that there is compensation by negatively curved neighbours. To this end, let $c^*(\hat{\Delta})$ denote the sum of $c(\hat{\Delta})$ together with all possible additions of $c(\Delta)$, where $c(\Delta) > 0$ and Δ shares an edge with $\hat{\Delta}$; this will be made more precise later. The aim will be to show that $c^*(\hat{\Delta}) \leq 0$ for each non-distinguished negatively curved region $\hat{\Delta}$ of Γ, and then, in the spherical case, to show that $c^*(\hat{\Delta}) < 4\pi$ for the distinguished region $\hat{\Delta}$.

4. Quasi-asphericity

As in the last section, let G be the group defined by the presentation

$$\wp = \langle a, b : a^l = b^m = (ab)^n = (ab^{-1})^k = 1 \rangle,$$

and let a dipole be a connected spherical picture over $\wp$ containing exactly two vertices; an easy check shows that these must either both be α-vertices or both β-vertices. If we regard $\wp$ as a two-dimensional CW-complex Z, then $\pi_1(Z)$ is isomorphic to G and the spherical pictures we consider here represent elements of $\pi_2(Z)$ (see [16]). Note that bridge moves do not change the free homotopy class.

The presentation $\wp$ is *quasi-aspherical* over $C_l * C_m$ if every spherical picture over $\wp$ containing at least one vertex fails to be reduced. It follows that, if $\wp$ is quasi-aspherical, then, as a $\mathbb{Z}\pi_1(Z)$-module, $\pi_2(Z)$ is generated by dipoles.

We now prove:

Theorem 4.1. *If any of the following conditions hold, then $\wp$ is quasi-aspherical over $C_l * C_m$:*

(1) $m \geq l = 4, k \geq n \geq 4$;
(2) $m \geq l = 4, n = 3, k \geq 6$;
(3) $m \geq l \geq 5, k \geq n \geq 3$;
(4) $m \geq l \geq 7, n = 2, k \geq 5$;
(5) $m \geq l = 6, n = 2, k \geq 6$;
(6) $m > l = 5, n = 2, k \geq 6$;
(7) $l = 9, m \geq 11, n = 2, k \geq 3$;
(8) $l \geq 11, m \geq 11, n = 2, k \geq 3$;
(9) $l \geq 7, m \geq 8, n = 2, k = 4$.

Proof. In each case, suppose (by way of contradiction) that Π is a reduced spherical picture over $\wp$ containing at least one vertex and satisfying the condition that Π has the minimal number of vertices over all such pictures; in particular, Π is connected.

We form the graph Γ from Π as described in Section 3. In the following case-by-case analysis, we locate all possible positively curved regions Δ of Γ and indicate how to distribute $c(\Delta)$ to negatively curved regions $\hat{\Delta}$ to ensure that $c^*(\hat{\Delta}) \leq 0$ whenever $\hat{\Delta}$ is interior. This implies that, if $\hat{\Delta}$ is the distinguished region, then $c^*(\hat{\Delta}) \geq 4\pi$;

we obtain our contradiction at the end of the proof by showing that this cannot happen.

In the first five cases, we actually have that $c(\Delta) \le 0$ for any interior region Δ of Γ.

If $m \ge l = 4$ and $k \ge n \ge 4$, it follows that the degree of each vertex and each interior region of Γ is at least 4, and $c(\Delta) \le c(4,4,4,4) = 0$.

If $m \ge l = 4$, $n = 3$ and $k \ge 6$, then each interior region Δ has degree at least 4; if $c(\Delta) > 0$, then Δ must contain an α-vertex of degree 3; since such a vertex is adjacent only to β-vertices, it follows that $c(\Delta) \le c(3,3,6,6) = 0$.

Suppose that $m \ge l \ge 5$ and $k \ge n \ge 3$. If Δ has degree 5, then Δ either contains a pair of adjacent α-vertices or a pair of adjacent β-vertices, and $c(\Delta) \le c(3,3,3,4,4) = 0$. If Δ has degree 4, then Δ must be an $\tilde{A}$-region, and it follows from the comments made in Section 3 that $c(\Delta) \le c(4,4,4,4) = 0$.

If $m \ge l \ge 7$, $n = 2$ and $k \ge 5$, we again have that Δ has degree at least 4. If Δ has degree 5, then, as a region of Π, Δ must have contained at least two α-vertices of degree two, and so must contain at least three β-vertices, and so $c(\Delta) \le c(3,3,5,5,5) < 0$. If Δ has degree 4 and Δ is an $\tilde{A}$-region, then $c(\Delta) \le c(3,3,6,6) = 0$, and, if Δ has degree 4 and Δ is not an $\tilde{A}$-region, then Δ must contain four β-vertices and $c(\Delta) \le c(5,5,5,5) < 0$.

Suppose that $m \ge l \ge 6$, $n = 2$ and $k \ge 6$. If Δ is an $\tilde{A}$-region, then we again have that $c(\Delta) \le c(3,3,6,6) = 0$; so assume that Δ is not an $\tilde{A}$-region. If Δ has degree 5, then Δ contains at least two β-vertices, and $c(\Delta) \le c(3,3,3,6,6) < 0$. If Δ has degree 4, then $c(\Delta) \le c(3,6,6,6) < 0$, and, if Δ has degree 3, then $c(\Delta) \le c(6,6,6) = 0$.

In the remaining four cases, it turns out that Γ can contain interior regions Δ of positive curvature.

Suppose that $m > l = 5$, $n = 2$ and $k \ge 6$. If Δ is an $\tilde{A}$-region, then $c(\Delta) \le c(3,3,6,6)$; so assume that Δ is not an $\tilde{A}$-region. If Δ had contained in Π at least one α-vertex of degree two, then a similar argument to the previous case shows that $c(\Delta) \le 0$. It follows that, if $c(\Delta) > 0$, then Δ is given (up to cyclic permutation and inversion) by Fig. 7.

Observe that, in Fig. 7, α_i has degree three ($1 \le i \le 4$) and α_5 has degree three or four. We distribute

$$\tfrac{1}{3}c(\Delta) \le \tfrac{1}{3}c(3,3,3,3,3) = \tfrac{1}{9}\pi$$

to each of $c(\Delta_1)$, $c(\Delta_2)$ and $c(\Delta_3)$. This means that, if $\hat{\Delta}$ is a region of Γ that receives positive curvature from at least one neighbouring region, then $\hat{\Delta}$ is a B-region containing at least two β-vertices; a routine check now shows that $c^*(\hat{\Delta}) \le c(3,3,6,6,6) + \frac{1}{9}\pi < 0$.

Suppose that $l = 9$ or $l \ge 11$, and that $m \ge 11$, $n = 2$ and $k \ge 3$. If Δ is an interior region of Γ of degree less than 6, then either Δ is an $\tilde{A}$-region, and is given (up to cyclic permutation and inversion) by Fig. 8, or $l = 9$ and Δ is an A-region of degree 5.

If $l = 9$ and Δ is an A-region of degree 5, then Δ must either have two adjacent α-vertices, in which case Δ has degree at least 6, a contradiction, or two adjacent

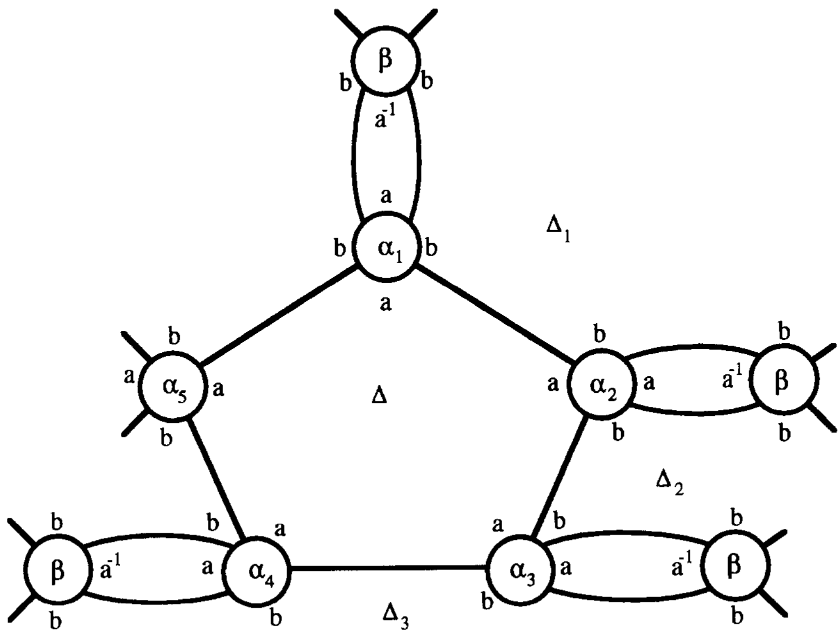


Fig. 7.

β-vertices, in which case $c(\Delta) \leq c(3,3,3,4,4) = 0$. So we need only consider the situation shown in Fig. 8.

In Figs.8(a) and (b), we distribute $\frac{1}{4}c(\Delta) \leq \frac{1}{4}c(3,3,4,4) = \frac{1}{12}\pi$ to each of $c(\Delta_i)$ $(1 \leq i \leq 4)$. In Fig. 8(c), we distribute $\frac{1}{2}c(\Delta) \leq \frac{1}{6}\pi$ to each of $c(\Delta_1)$ and $c(\Delta_2)$. As before, if $\hat{\Delta}$ is a region of Γ that receives positive curvature from at least one neighbouring region, then $\hat{\Delta}$ is a B-region and has degree at least 6. If $\hat{\Delta}$ has degree at least 12, then

$$c^*(\hat{\Delta}) \leq [-10 + 12(\tfrac{2}{3}) + 12(\tfrac{1}{6})]\pi = 0,$$

and so we may assume that $6 \leq \deg(\hat{\Delta}) \leq 11$. If $\hat{\Delta}$ has degree $6+\theta$, where $0 \leq \theta \leq 5$, then $\hat{\Delta}$ in Π contained $5-\theta$ α-vertices of degree 2, which means that $\hat{\Delta}$ can receive positive curvature across at most $2\theta+1$ edges, in which case either the degree of each vertex in $\hat{\Delta}$ is 3, in which case, since both β-vertices in Fig. 8(c) have degree greater than 3, $\hat{\Delta}$ receives at most $\frac{1}{12}\pi$ across each edge, and

$$c^*(\hat{\Delta}) \leq [-(4+\theta) + \tfrac{2}{3}(6+\theta) + \tfrac{1}{12}(2\theta+1)]\pi < 0,$$

or $\hat{\Delta}$ has at least one vertex of degree greater than 3 and

$$c^*(\hat{\Delta}) \leq [-(4+\theta) + \tfrac{2}{3}(5+\theta) + \tfrac{2}{4} + \tfrac{1}{6}(2\theta+1)]\pi = 0.$$

Lastly, we consider the case where $l \geq 7$, $m \geq 8$, $n = 2$ and $k = 4$. Here, if Δ has degree 5, then $c(\Delta) \leq c(3,3,4,4,4) < 0$, and, if Δ is an A-region of degree 4, then

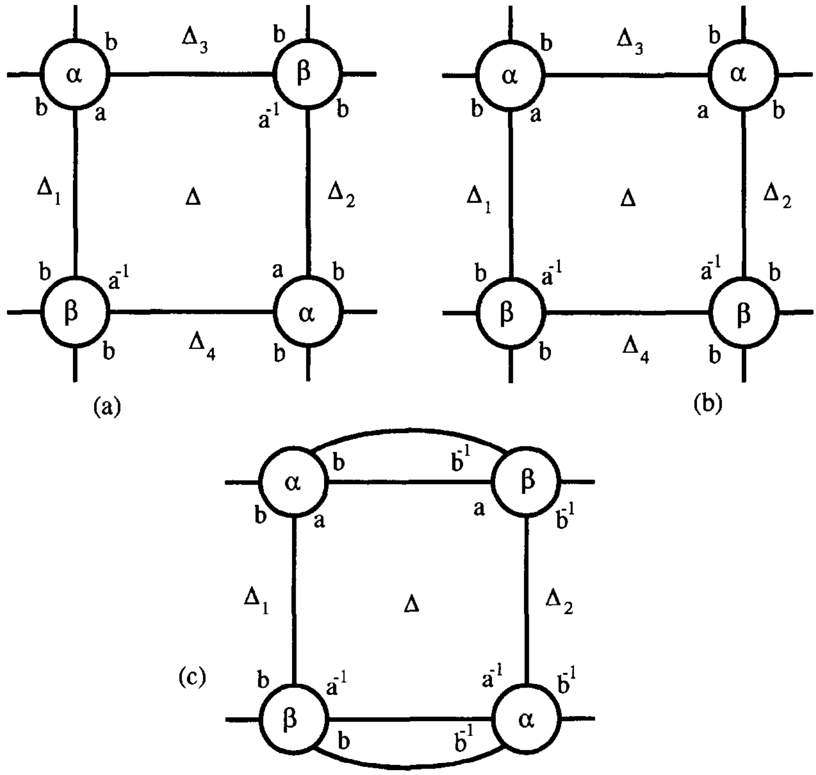


Fig. 8.

$c(\Delta) \le c(4,4,4,4) = 0$. Therefore, as in the previous case, the regions of Γ of positive curvature are given by Fig. 8; note, however, in this case that $c(\Delta) \le c(3,3,5,5) = \frac{2}{15}\pi$. So, if $\hat{\Delta}$ receives positive curvature, then $\hat{\Delta}$ is a B-region and has degree at least five. If $\hat{\Delta}$ has degree five, then $\hat{\Delta}$ must have contained three α-vertices of degree two in Π, and so

$$c^*(\hat{\Delta}) \le c(3,3,4,4,4) + \tfrac{2}{15}\pi < 0;$$

if Δ has degree 6, then

$$c^*(\hat{\Delta}) \le c(3,3,3,3,4,4) + \tfrac{4}{15}\pi < 0;$$

if Δ has degree 7, then

$$c^*(\hat{\Delta}) \le c(3,3,3,3,3,3,4) + \tfrac{6}{15}\pi < 0;$$

lastly, if Δ has degree at least 8, then

$$c^*(\hat{\Delta}) \le -6 + 8(\tfrac{2}{3}) + \tfrac{8}{15}\pi < 0.$$

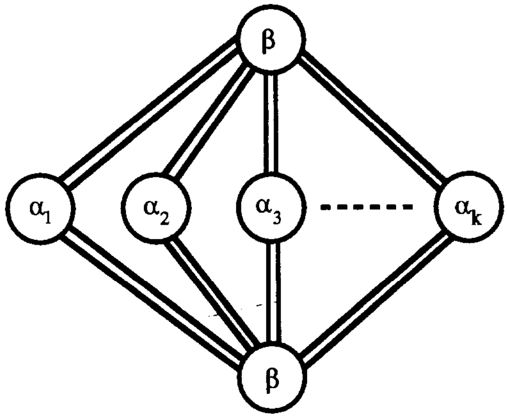


Fig. 9.

It follows from the above that $\frac{1}{6}\pi$ is the maximum amount of curvature transferred across any edge. Therefore, if $\hat{\Delta}$ is the distinguished region and $\hat{\Delta}$ has degree q, then

$$c^*(\hat{\Delta}) \leq [(2-q) + q(\tfrac{2}{3}) + q(\tfrac{1}{6})]\pi < 4\pi,$$

and this yields the contradiction. □

It turns out that there are examples of non-trivial reduced spherical pictures over $\wp$ for certain values of l, m, n and k, examples of which are shown in Figs. 9 and 10. Let D denote the set of elements of $\pi_2(Z)$ represented by dipoles. With this notation, we have:

Theorem 4.2. (1) *If $l = 4$, $n = 2$, and if either $m \geq 7$, $k \geq 7$, or $m \geq 12$, $k = 5$, then $\pi_2(Z)$ is generated by $D \cup \mathscr{S}_1$, where $\mathscr{S}_1$ is given by Fig.* 9.

(2) *If $l = 6$, $m \geq 13$, $n = 2$ and $k = 3$, then $\pi_2(Z)$ is generated by $D \cup \mathscr{S}_2$, where $\mathscr{S}_2$ is given by Fig.* 10.

(3) *If $l = 4$, $m \geq 13$, $n = 2$ and $k = 3$, then $\pi_2(Z)$ is generated by $D \cup \mathscr{S}_3$, where $\mathscr{S}_3$ is given by Fig.* 9 *with $k = 3$.*

Proof. Suppose, by way of contradiction, that $\pi_2(Z)$ is not generated by $D \cup \mathscr{S}_i$ (where $i = 1, 2$ or 3 as appropriate), so that there exist pictures over $\wp$ representing elements of $\pi_2(Z) - \langle D \cup \mathscr{S}_i \rangle$. Among all such pictures, choose one, Π say, with the minimal number of vertices; this ensures, for example, that Π is reduced and connected.

We obtain Γ from Π as before, except that there are differences, which we now discuss. In what follows, we refer to the three situations described in the statement of our result as Case 1, Case 2 and Case 3.

First observe that, according to case, Π cannot contain a subpicture of Fig. 9 or 10 which contains more than half the vertices (otherwise we could reduce the number of vertices in Π, contradicting the minimality condition). This means, for example, that, in Case 1, if Π contains a subpicture of the form shown in Fig. 11, then there are at

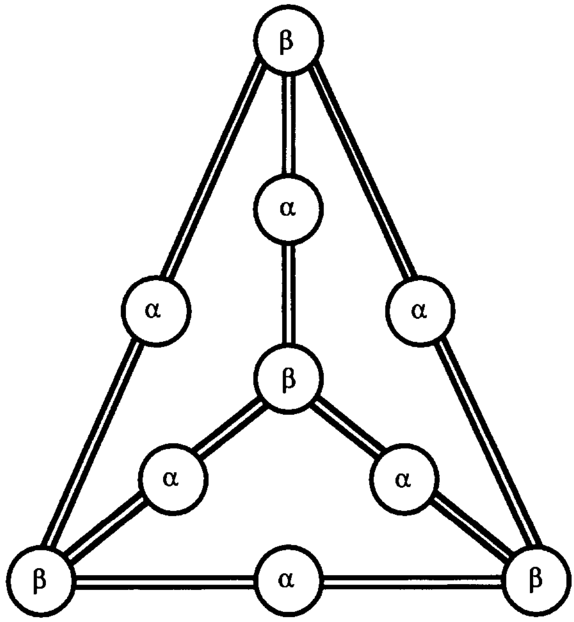


Fig. 10.

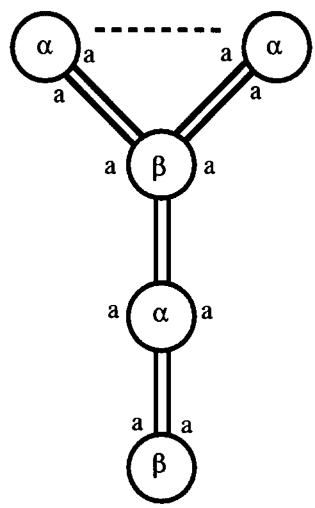


Fig. 11.

most $\frac{1}{2}(k-2)$ α-vertices if k is even, and at most $\frac{1}{2}(k-3)$ α-vertices if k is odd. This, in turn, implies that, if $k = 3$, then Π has no α-vertices of degree 2 of the type shown in Fig. 5. Another consequence is that, in Case 2, Γ will not contain any A-region of positive curvature. A further difference for Case 1 is that Γ may contain parallel edges between β-vertices which arise in the way shown in Fig. 12; in this case, we simply identify any parallel edges as illustrated there.

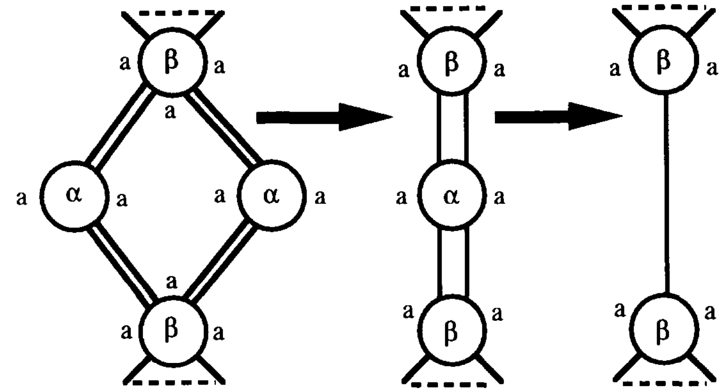


Fig. 12.

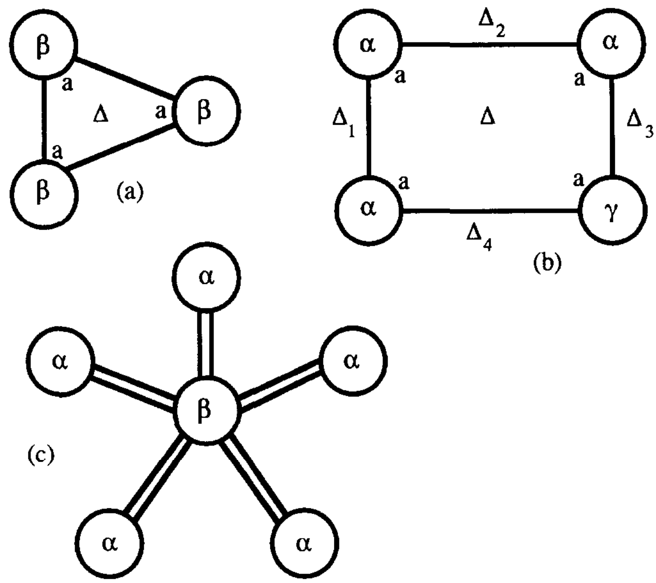


Fig. 13.

In Case 1 with $m \geq 7$ and $k \geq 7$, then, since we have $k \geq 7$, it follows from the statements made above that every β-vertex has degree at least six; therefore, if Δ is an $\tilde{A}$-region of Γ, then $c(\Delta) \leq c(3,3,6,6) = 0$. If Δ is an interior region of degree 5, then, since $m \geq 7$, Δ must have contained in Π two α-vertices of degree 2, and $c(\Delta) \leq c(3,3,6,6,6) < 0$. If Δ has degree 3, then Δ is given by Fig. 13(a), and $c(\Delta) \leq c(6,6,6) = 0$. So, if $c(\Delta) > 0$, then Δ is given (up to cyclic permutation and inversion) by Fig. 13(b) (in which γ is either α or β).

In Fig. 13(b), if $\gamma = \beta$, then add $\frac{1}{2}c(\Delta) \leq \frac{1}{2}c(3,3,3,6) = \frac{1}{6}\pi$ to $c(\Delta_1)$ and $c(\Delta_2)$; if $\gamma = \alpha$, then it can be assumed, without any loss of generality, that α_1 has degree 3, in which case we add $\frac{1}{6}\pi$ from $c(\Delta)$ to each of $c(\Delta_1)$ and $c(\Delta_2)$, and, if necessary, $\frac{1}{2}[c(\Delta) - \frac{1}{3}\pi]$ to each of $c(\Delta_3)$ and $c(\Delta_4)$. This means that positive curvature is distributed only across edges that join α-vertices, and, if this amount (which is at most $\frac{1}{6}\pi$) exceeds $\frac{1}{12}\pi$, then at least one of the α-vertices is adjacent in the region receiving the positive curvature to a β-vertex. It follows that, if $\hat{\Delta}$ is an interior region of Γ that receives positive curvature, then $\hat{\Delta}$ is a B-region of degree at least 5 containing at least one β-vertex. If $\hat{\Delta}$ has degree 5, then

$$c^*(\hat{\Delta}) \le c(3,3,6,6,6) + \tfrac{1}{6}\pi < 0,$$

if $\hat{\Delta}$ has degree 6, then

$$c^*(\hat{\Delta}) \le c(3,3,3,3,6,6) + \tfrac{5}{12}\pi < 0,$$

and, if $\hat{\Delta}$ has degree at least 7, then

$$c^*(\hat{\Delta}) \le c(3,3,3,3,3,3,6) + \tfrac{7}{12}\pi < 0.$$

In Case 1 with $m \ge 12$ and $k = 5$, there can occur a β-vertex of degree less than 6, and this is shown in Fig. 13(c). Since $m \ge 12$ and there are no α-vertices of degree 2 in Π, it follows that, if $c(\Delta) > 0$, then Δ must be an $\tilde{A}$-region or an A-region of degree 4. We distribute $c(\Delta)$ uniformly among the neighbouring B-regions of Δ; it follows that, if $\hat{\Delta}$ receives positive curvature, then $\hat{\Delta}$ is a B-region of degree at least 12, and

$$c^*(\hat{\Delta}) \le [-10 + 12(\tfrac{2}{3}) + 12(\tfrac{1}{6})]\pi = 0.$$

For Case 2, if $c(\Delta) < 0$, then Δ is one of the $\tilde{A}$-regions of Fig. 8. We distribute $c(\Delta) \le c(3,3,4,4) = \frac{\pi}{3}$ uniformly among the neighbouring B-regions. Let $\hat{\Delta}$ be an interior region of Γ that receives positive curvature. If $\hat{\Delta}$ has degree at least 12, then

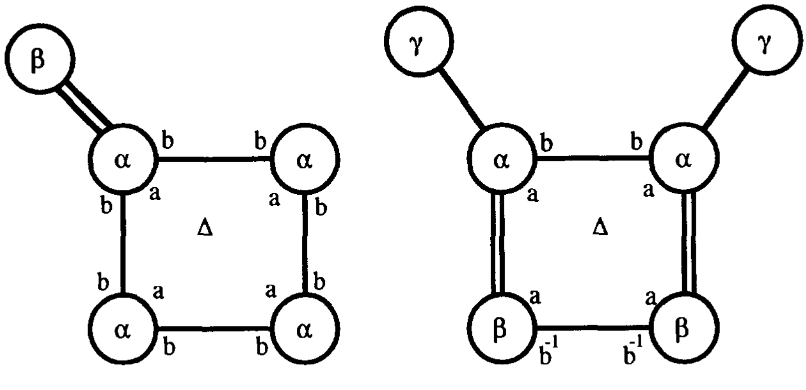
$$c^*(\hat{\Delta}) \le [-10 + 12(\tfrac{2}{3}) + 12(\tfrac{1}{6})]\pi = 0,$$

and, if $7 \le \theta = \deg(\hat{\Delta}) \le 11$, then

$$c^*(\hat{\Delta}) \le [-(\theta - 2) + \tfrac{2}{3}\theta + \tfrac{1}{6}(2\theta - 13)]\pi < 0.$$

For Case 3, observe that, since Π has no subpicture containing more than half of $\mathcal{S}_3$, it follows that the degree of any β-vertex contained in an A-region is at least 4. Therefore, if $c(\Delta) > 0$, then either Δ is one of the $\tilde{A}$-regions of Fig. 8, or Δ is one of the regions of Fig. 14. We distribute $c(\Delta)$ uniformly to the neighbouring B-regions. In Fig. 14(a), each B-region will receive at most $\frac{1}{4}c(3,3,3,3) = \frac{1}{6}\pi$, and, in Fig. 14(b), each B-region will receive at most $\frac{1}{2}c(3,3,4,4) = \frac{1}{6}\pi$. If $\hat{\Delta}$ is an interior region of Γ that receives positive curvature then, since $m \ge 13$, the proof that $c^*(\Delta) \le 0$ is similar to that for Case 2.

So we have now shown that, for all cases, if $\hat{\Delta}$ is an interior region of Γ that receives positive curvature, then $c^*(\hat{\Delta}) \le 0$, and that $\frac{1}{6}\pi$ is the maximum amount of curvature transferred across any edge. A contradiction is now obtained in the same way as in Theorem 4.1. □



(a) (b)

Fig. 14.

We now turn to the groups $(10,m\,|\,2,3)$, $(6,m\,|\,2,5)$ and $(4,m\,|\,3,5)$ with presentations

$$\langle a,b : a^{10} = b^m = (ab)^2 = (ab^{-1})^3 = 1\rangle,$$
$$\langle a,b : a^6 = b^m = (ab)^2 = (ab^{-1})^5 = 1\rangle,$$
$$\langle a,b : a^4 = b^m = (ab)^3 = (ab^{-1})^5 = 1\rangle,$$

respectively. We introduce new generators $u = ab$ and $v = b^{-1}a$, and then delete $a = bv$, to get the presentations

$$\langle u,v,b : u^2 = v^3 = (uv)^5 = b^m = u^{-1}bvb = 1\rangle,$$
$$\langle u,v,b : u^2 = v^5 = (uv)^3 = b^m = u^{-1}bvb = 1\rangle,$$
$$\langle u,v,b : u^3 = v^5 = (uv)^2 = b^m = u^{-1}bvb = 1\rangle.$$

We see that each group is a one-relator product of $A \cong A_5$ and $B \cong C_m$, with the extra relator being $u^{-1}bvb$, and we work with spherical pictures over this product. This is similar to the two-relator case, except that each corner label will now be one of $\{u, u^{-1}, v, v^{-1}, b, b^{-1}\}$ and reading round any vertex (in the clockwise direction) yields $(u^{-1}bvb)^{\pm 1}$ as a cyclically reduced word in $A * B$. Furthermore, it is clear that if the picture is reduced, then the label of any A-region will be cyclically reduced; by making moves similar to the one shown in Fig. 4, it can be assumed that the label of any B-region does not contain a substring bb^{-1} or $b^{-1}b$. As before, we can regard each presentation as a two-dimensional CW-complex Z with $\pi_1(Z)$ isomorphic to the corresponding group, and we let D denote the set of elements of $\pi_2(Z)$ represented by dipoles.

Theorem 4.3. (1) *If $l = 10$, $m \geq 15$, $n = 2$ and $k = 3$, then $\pi_2(Z)$ is generated by $D \cup \mathcal{T}_1$, where $\mathcal{T}_1$ is given by Fig.* 15 *with $r = 10$.*

(2) *If $l = 6$, $m \geq 9$, $n = 2$ and $k = 5$, then $\pi_2(Z)$ is generated by $D \cup \mathcal{T}_2$, where $\mathcal{T}_2$ is given by Fig.* 15 *with $r = 6$.*

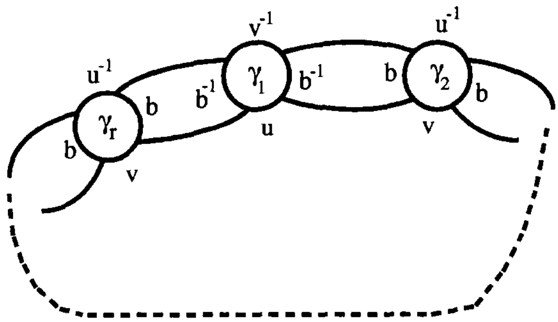


Fig. 15.

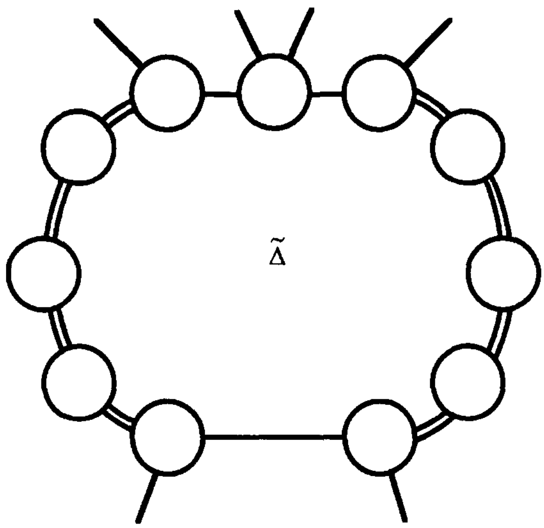


Fig. 16.

(3) *If* $l = 4$, $m \geq 11$, $n = 3$ *and* $k = 5$, *then* $\pi_2(Z)$ *is generated by* $D \cup \mathcal{T}_3$, *where* $\mathcal{T}_3$ *is given by Fig.* 15 *with* $r = 4$.

Proof. As in the proof of Theorem 4.2, we assume, by way of contradiction, that $\pi_2(Z)$ is not generated by $D \cup \mathcal{T}_i$ (where $i = 1, 2$ or 3 as appropriate), and we let Π be a counterexample having the minimum number of vertices.

We form the graph Γ from Π in a similar way to the method discussed in Section 3, and let Δ be an interior region of Γ of positive curvature obtained from the region $\tilde{\Delta}$ of Π.

If $l = 10$, $m \geq 15$, $n = 2$ and $k = 3$, then it follows from the facts that Δ has degree at most five and that Π cannot contain more than half of $\mathcal{T}_1$ that $\tilde{\Delta}$ has degree at most 11. In fact, the maximal case is shown in Fig. 16.

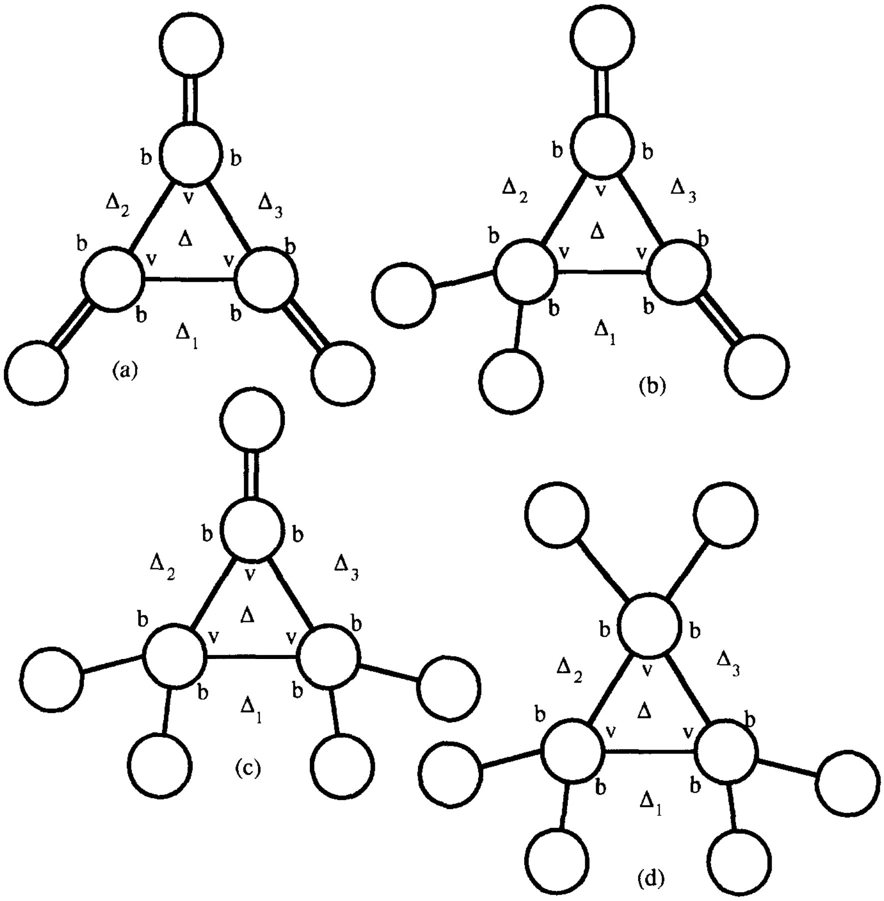


Fig. 17.

Since $\tilde{\Delta}$ is an A-region, it follows that we need to check cyclically reduced words in u and v of length at most 11. (Recall that Π is reduced.) Such a check shows that, in fact, Δ does not have degree two or four, and, if Δ has degree five, then at least three of the edges of Δ are contained in adjacent B-regions. If Δ has degree five, then distribute $\frac{1}{3}c(\Delta) \leq \frac{1}{9}\pi$ to each of the B-regions across the appropriate edges. If Δ has degree three, then Δ is given (up to cyclic permutation and inversion) by Figs. 17(a)–(d).

In Fig. 17(a), distribute $\frac{1}{3}c(\Delta) = \frac{1}{3}\pi$ to each of $c(\Delta_i)$, $1 \leq i \leq 3$. In Fig. 17(b), $c(\Delta) = \frac{5}{6}\pi$, so distribute $\frac{1}{3}\pi$ to each of $c(\Delta_1)$ and $c(\Delta_2)$ and $\frac{1}{6}\pi$ to $c(\Delta_3)$. In Fig. 17(c), $c(\Delta) = \frac{2}{3}\pi$, so distribute $\frac{1}{3}\pi$ to $c(\Delta_1)$ and $\frac{1}{6}\pi$ to each of $c(\Delta_2)$ and $c(\Delta_3)$. In Fig. 17(d), distribute $\frac{1}{3}c(\Delta) = \frac{1}{6}\pi$ to each of $c(\Delta_i)$, $1 \leq i \leq 3$.

If $l = 6$, $m \geq 9$, $n = 2$ and $k = 5$, then it is easy to verify (using the fact that Π does not contain more than half of $\mathcal{T}_2$) that, if $c(\Delta) > 0$, then Δ has degree 5,

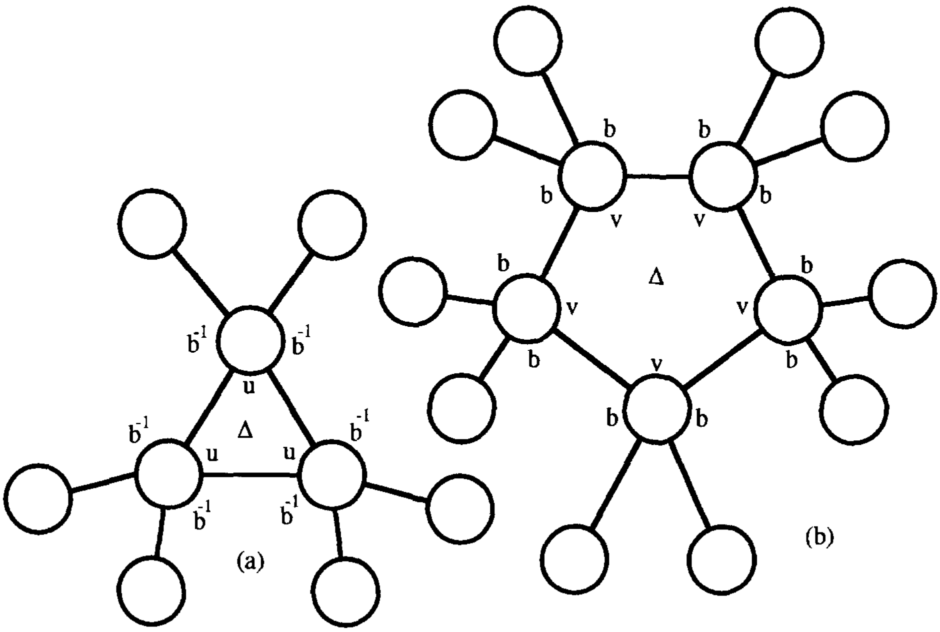


Fig. 18.

and, again, there are at least three edges of Δ contained by B-regions adjacent to Δ. Distribute $\frac{1}{3}c(\Delta) \leq \frac{1}{9}\pi$ to each of these B-regions.

If $l = 4$, $m \geq 11$, $n = 3$ and $k = 5$, then it is easy to verify (using the fact that Π does not contain more than half of $\mathcal{T}_3$, and so, in fact, no vertices of degree two) that Δ is given (up to cyclic permutation and inversion) by Figs. 18(a) and (b).

In Figs. 18(a), distribute $\frac{1}{3}c(\Delta) = \frac{1}{6}\pi$ to each of $c(\Delta_i)$, $1 \leq i \leq 3$. In Fig. 18(b), distribute $\frac{1}{5}c(\Delta) \leq \frac{1}{15}\pi$ to each of $c(\Delta_j)$, $1 \leq j \leq 5$.

We have described, in each case, the interior regions of positive curvature and the distribution of such curvature. Now let $\hat{\Delta}$ be an interior region of Γ that receives curvature across at least one edge. Let $\hat{\Delta}$ have degree θ.

If $l = 10$, $m \geq 15$, $n = 2$ and $k = 3$, the key observation is that, if $\frac{1}{3}\pi$ is transferred across any edge e, then, as can be seen in Fig. 17, the edge immediately to the left of e belongs to two B-regions, and so no curvature is transferred across it. It follows that

$$c^*(\hat{\Delta}) \leq \left[-(\theta - 2) + \frac{2\theta}{3} + \frac{\theta}{6}\right]\pi,$$

and, since $\theta \geq 15$, we obtain $c^*(\hat{\Delta}) \leq 0$.

If $l = 6$, $m \geq 9$, $n = 2$ and $k = 5$, then

$$c^*(\hat{\Delta}) \leq \left[-(\theta - 2) + \frac{2\theta}{3} + \frac{\theta}{9}\right]\pi,$$

and, since $\theta \geq 9$, it follows that $c^*(\hat{\Delta}) \leq 0$.

If $l = 4$, $m \geq 11$, $n = 3$ and $k = 5$, and $\hat{\Delta}$ does not receive $\frac{1}{6}\pi$ across any edge, then

$$c^*(\hat{\Delta}) \leq \left[-(\theta - 2) + \frac{2\theta}{3} + \frac{\theta}{15}\right]\pi.$$

Otherwise, it is clear from Fig. 18(a) that $\hat{\Delta}$ contains at least two vertices of degree 4, and

$$c^*(\hat{\Delta}) \leq \left[-(\theta - 2) + \frac{2(\theta - 1)}{3} + \frac{2}{4} + \frac{\theta}{6}\right]\pi.$$

Since $\theta \geq 11$, it follows that $c^*(\hat{\Delta}) \leq 0$.

Finally, if $\hat{\Delta}$ is the distinguished region, then it is clear from the above that

$$c^*(\hat{\Delta}) \leq \left[-(\theta - 2) + \frac{2\theta}{3} + \frac{\theta}{6}\right]\pi < 4\pi,$$

and this yields the desired contradiction. □

5. Non-collapsing

As in Section 2, we say that the group $(l,m\,|\,n,k)$ defined by the presentation

$$\wp = \langle a,b : a^l = b^m = (ab)^n = (ab^{-1})^k = 1\rangle$$

does not collapse if a, b, ab and ab^{-1} have orders l, m, n and k, respectively. We now prove:

Theorem 5.1. *If l, m, n and k satisfy any of the hypotheses of Theorem 4.1, Theorem 4.2 or Theorem 4.3, then $(l,m\,|\,n,k)$ does not collapse.*

Proof. It follows from Theorems 4.1 and 4.2 that no spherical picture over $\wp$ has boundary label a non-trivial element of $\langle a : a^l = 1\rangle$ or $\langle b : b^m = 1\rangle$; so a and b have orders l and m, respectively; it remains to show that ab has order n and ab^{-1} has order k.

Consider first the situation where one of the hypotheses of Theorem 4.1 is satisfied. Suppose, by way of contradiction, that ab has order s where $s < n$, so that $n = sr$ for some $r > 1$. Let Π_1 be a picture with boundary label $(ab)^s$; then the disjoint union of r copies of Π_1 has boundary label $(ab)^n$. We can form a spherical picture Π from this by adding a single α-vertex labelled $(ab)^{-n}$; thus, the number of α-vertices in Π labelled α minus the number of α-vertices labelled α^{-1} in Π is congruent to -1 (mod r), and is therefore non-zero (as $r > 1$). This contradicts the fact that $\pi_2(Z)$ is generated by dipoles, and any dipole containing an α-vertex contains exactly one of each sign. So ab has order n. A similar argument shows that ab^{-1} has order k.

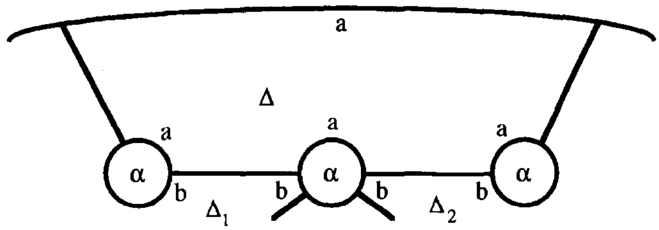


Fig. 19.

So we turn to the case where one of the hypotheses of Theorem 4.2 is satisfied. In all but the case where $l = 4$, $n = 2$, $m \geq 7$ and $k \geq 7$, the result is immediate, since ab having order less than n or ab^{-1} having order less than k would force b to have order less than m, contradicting the above. So we have this last case to consider, and it is clear that ab must have order 2 (else b would again have order less than m).

Suppose then that we have a picture Π over $\wp$ with boundary label $(a^{-1}b)^u$, where $0 < u < k$, and assume that the sum of the number of α-vertices and the number of β-vertices of Π is minimal over all such pictures. If Δ is a boundary region of Π of positive curvature, then (up to cyclic permutation and inversion) Δ is as in Fig. 19.

We distribute $\frac{1}{2}c(\Delta) \leq \frac{1}{2}c(3,3,3,3,3) = \frac{1}{6}\pi$ to each of Δ_1 and Δ_2. If Δ is an interior region of Π of positive curvature, then distribute $c(\Delta)$ as described in the proof of Theorem 4.2. Let $\hat{\Delta}$ be any region of Π that receives positive curvature. It follows that it is still the case that positive curvature (of at most $\frac{1}{6}\pi$) is distributed only across vertices joining α-vertices, and, if the amount exceeds $\frac{1}{12}\pi$, then at least one of the α-vertices is adjacent in $\hat{\Delta}$ to a β-vertex. The argument that $c^*(\hat{\Delta}) \leq 0$ is now similar to the one in Theorem 4.2. This contradiction yields the result.

Lastly, suppose that l, m, n and k satisfy one of the hypotheses of Theorem 4.3. It follows from Theorem 4.3 that no spherical picture over the one-relator product $A_5 * C_m/\overline{\{u^{-1}bvb\}}$ has boundary label a non-trivial element of $\langle b : b^m = 1 \rangle$, and so b has order m. If uv does not have the prescribed order in any of these cases, then $uv = 1$, so that $u = v = 1$, and then $b = a$, which gives $b^2 = 1$, a contradiction. So uv has the prescribed order, and so $a^2 = uv$ has order $\frac{l}{2}$ (equal to 5, 3 or 2 according to case). If a has order l, then we have proved the theorem, since any collapse of the order of ab or ab^{-1} clearly forces $a = b^{\pm 1}$, a contradiction. So assume that a has order $\frac{l}{2} = 5, 3$ or 2. We thus have the group $(5, m \mid 2, 3)$, $(3, m \mid 2, 5)$ or $(2, m \mid 3, 5)$. The group $(5, m \mid 2, 3)$ is either isomorphic to A_5 or is trivial by Proposition 2.1, $(3, m \mid 2, 5)$ is isomorphic to $(3, 2 \mid m, 5)$ (see the comment after Proposition 1.1), i.e. $(2.3 \mid 5, m)$, which is isomorphic to A_5 or is trivial by Proposition 1.1, and $(2, m \mid 3, 5)$ is trivial by Proposition 1.1. So, in all cases, we either have A_5 or the trivial group, so that b has order at most 5, a contradiction. Thus a has order l as required. □

Let

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

be a pushout of groups. If K_A, K_B and K_C are Eilenberg–Maclane spaces of types $K(A,1)$, $K(B,1)$ and $K(C,1)$, respectively, and if $\hat{\psi}: K_A \to K_B$ and $\hat{\phi}: K_A \to K_C$ denote the continuous maps realizing ψ and ϕ at the fundamental group level, then we can form a space X with $\pi_1(X) = D$ by setting $X = M(\hat{\psi}) \cup_{K_A} M(\hat{\phi})$, where $M(\cdot)$ denotes the mapping cylinder. We say that the pushout is *geometrically Mayer–Vietoris* if X is aspherical, i.e. a $K(D,1)$ space. If this is the case, then the following two facts can be deduced (see [16, Section 5; 8, Section 4]):

1. The (co)homology of A, B, C and D with coefficients in a given RD-module (where R is any commutative ring with identity) is linked by a Mayer–Vietoris sequence.
2. If A, B and C is each of type $FP_{\mathbb{Q}}$, then so is D, and, moreover,

$$\chi_{\mathbb{Q}}(D) = \chi_{\mathbb{Q}}(B) + \chi_{\mathbb{Q}}(C) - \chi_{\mathbb{Q}}(A),$$

where $\chi_{\mathbb{Q}}$ is the rational Euler characteristic.

We may now prove the following.

Theorem 5.2. *If any of the hypotheses of Theorem* 5.1 *hold, then the group* $G = (l,m \mid n,k)$ *is infinite.*

Proof. If l, m, n and k satisfy one of the hypotheses of Theorem 4.3, then G is infinite by Proposition 2.9.

Next assume that l, m, n and k satisfy one of the hypotheses of Theorem 4.1. If we take

$$A = \langle c,d : \ \rangle, \ B = \langle c,d : c^n = d^k = 1\rangle, \text{ and } C = \langle a,b : a^l = b^m = 1\rangle,$$

and if we define $\psi : A \to B$ by $c\psi = c$, $d\psi = d$, and $\phi : A \to C$ by $c\phi = ab$, $d\phi = ab^{-1}$, then D in the pushout is isomorphic to G. Since G does not collapse by Theorem 5.1, we can use an argument similar to that used in Section 5 of [7] to show that the pushout is geometrically Mayer–Vietoris, in which case, if D were finite, we would have

$$\mathrm{Hom}(D,M) \cong \mathrm{Hom}(C_l,M) \times \mathrm{Hom}(C_m,M) \times \mathrm{Hom}(C_n,M) \times \mathrm{Hom}(C_k,M),$$

where, in particular, M can be taken to be the cyclic group of order $lmnk$. Now it is clear that

$$|\mathrm{Hom}(G,C_{lmnk})| \le |\mathrm{Hom}(C_l,C_{lmnk})| \times |\mathrm{Hom}(C_m,C_{lmnk})|,$$

since, once we have specified the images of a and b, we have no choice with ab and ab^{-1}. So we can only have the isomorphism stated if $\mathrm{Hom}(C_n, C_{lmnk})$ and $\mathrm{Hom}(C_k, C_{lmnk})$ are trivial, a contradiction. So D is infinite, and hence G is infinite.

Now assume that $l = 6$, $m \geq 13$, $n = 2$ and $k = 3$. This time, we let

$$A = \langle c, d : c^3 = d^3 = 1 \rangle, \; B = \langle c, d : c^3 = d^3 = (cd)^2 = 1 \rangle,$$
$$\text{and } C = \langle a, b : a^6 = b^m = (a^{-1}b)^3 = 1 \rangle,$$

and define $\psi : A \to B$ by $d\psi = d$, $c\psi = c$, and $\phi : A \to C$ by $c\phi = a^2$, $d\phi = ba^{-1}$; then the group D is isomorphic to G. Note that B is isomorphic to A_4 and that C is the triangle group $(3, 6, m)$. The non-collapsing of G (Theorem 5.1) shows that the kernel of the map from A to G is free, and hence of homological dimension one, that the kernel of the map from B to G is trivial, and that the kernel of the map from C to G is a torsion-free surface group, and so has homological dimension at most two. If we can show further that, for the $K(D, 1)$ space X mentioned above, $\pi_2(X) = 0$, then it follows from Theorem 4.2 of [15] that X is aspherical, and hence that the pushout is geometrically Mayer–Vietoris. If this is the case, and if D has finite order N, then applying our fact about Euler characteristics yields

$$\frac{1}{N} = \frac{1}{12} + \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{m} - 1\right) - \left(\frac{1}{3} + \frac{1}{3} - 1\right) = \frac{1}{m} - \frac{1}{12},$$

which contradicts the fact that $m \geq 13$. So $D \cong (6, m \mid 2, 3)$ is infinite.

To show that $\pi_2(X) = 0$, we need only show that the generators of $\pi_2(X)$ come from the presentations for B and C. This is clear for dipoles, and so we need only consider $\mathscr{S}_2$. Note that the identities $d = ba^{-1}$, $c = a^2$ change $(ab)^2$ to $(cd)^2$ and $(ab^{-1})^3$ to d^3. It follows that, if we make these identifications, then $\mathscr{S}_2$ transforms to $\mathscr{S}_2'$ of Fig. 20(a). But $\mathscr{S}_2'$ corresponds to the spherical van Kampen diagram of Fig. 20(b), and hence is zero in $\pi_2(K_B)$. This shows that $\pi_2(X) = 0$ as required.

Suppose that $l = 4$, $m \geq 13$, $n = 2$ and $k = 3$. Let

$$A = \langle c, d : c^2 = d^2 = 1 \rangle, \qquad B = \langle c, d : c^2 = d^2 = (cd)^3 = 1 \rangle$$
$$\text{and} \quad C = \langle a, b : a^4 = b^m = (ab)^2 = 1 \rangle,$$

and define $\psi : A \to B$ by $c\psi = c$, $d\psi = d$, and $\phi : A \to C$ by $c\phi = ab$, $d\phi = a^2$; then D is isomorphic to G. A similar argument to the one used above shows that the pushout is geometrically Mayer–Vietoris (the corresponding transformed picture $\mathscr{S}_3$ and spherical van Kampen diagram are shown in Fig. 21).

If D has finite order N, then applying our fact about Euler characteristics yields

$$\frac{1}{N} = \frac{1}{6} + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{m} - 1\right) - \left(\frac{1}{2} + \frac{1}{2} - 1\right) = \frac{1}{m} - \frac{1}{12},$$

which again contradicts the fact that $m \geq 13$.

This only leaves the groups $(4, m \mid 2, k)$ with $m \geq 7$ and $k \geq 7$ or $m \geq 12$ and $k = 5$, where $1/m + 1/k > \frac{1}{4}$. In each case, we are considering the group G with

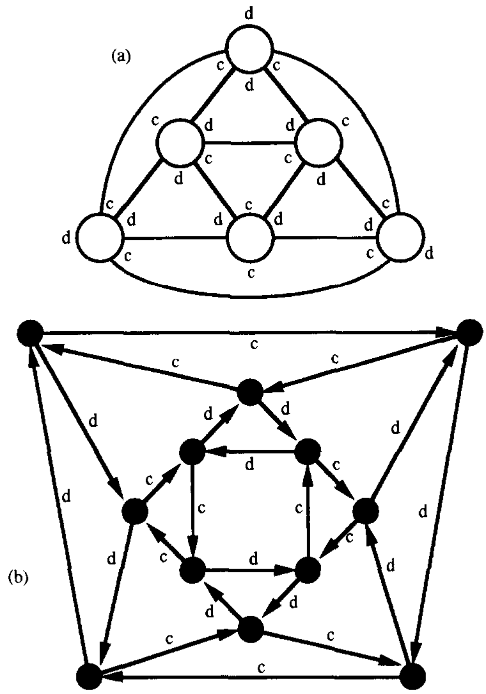


Fig. 20.

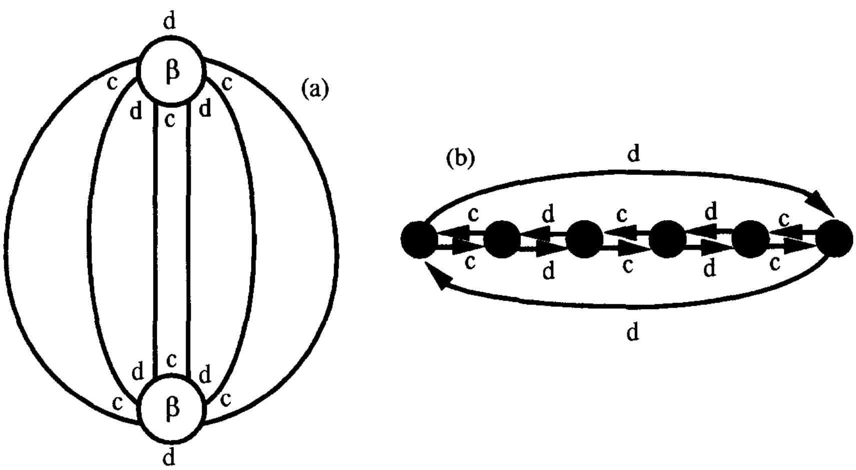


Fig. 21.

presentation

$$\langle a, b : a^4 = b^m = (ab)^2 = (ab^{-1})^k = 1 \rangle.$$

The groups $(4, m \mid 2, k)$ with m or k even are covered by Corollary 2.5; so we need to consider the groups $(4, m \mid 2, k)$ with m and k both odd.

$(4,7\,|\,2,7)$: We have a mapping from G to the alternating group A_{15} defined by

$$a \mapsto (2,3,4,5)(6,8,10,7)(9,12,13,11)(14,15),$$

$$b \mapsto (1,2,4,7,9,6,3)(8,11,13,15,14,12,10).$$

Let H be the subgroup generated by a, $b^2a^{-1}b$ and $b^{-2}ab^2ab^{-1}a^{-1}b^2$. Then H has index 15 in G and the presentation

$$\langle x,y,z : z^2 = x^4 = (xy^{-1})^7 = (x^2yx^{-1}y^2zy^{-1}zxy^{-1})^2 \\ = (x^2yx^{-1}yzxy^{-1})^7 = 1\rangle.$$

Since $\frac{1}{2}+\frac{1}{4}+\frac{1}{7}+\frac{1}{2}+\frac{1}{7} < 2$, H is infinite by Proposition 2.9 if there is no collapse. However, the mapping from G to A_{15} described above gives rise to a mapping from H to A_{15} defined by

$$x \mapsto (2,3,4,5)(6,8,10,7)(9,12,13,11)(14,15),$$

$$y \mapsto (2,6,15,10,8)(3,12,11,14,13,4,9,7,5),$$

$$z \mapsto (2,12)(4,9)(5,15)(6,7)(10,14)(11,13).$$

This shows that there is no collapse, so that H is infinite, and hence G is infinite.

$(4,9\,|\,2,7)$: If we let H be the subgroup of G generated by a, b^2ab^{-1}, $ba^{-1}ba^{-1}b^2$ and $b^{-1}ab^4$, then H has index 9 in G and H/H' is isomorphic to $C_7 \times C_\infty$, so that G is infinite.

$(4,7\,|\,2,9)$: We have an obvious homomorphism ϕ from $(4,7\,|\,2,9)$ to $(4,7\,|\,2,3)$, which is isomorphic to $PSL(2,7)$ by Proposition 2.1. If K is the kernel of ϕ, then K has a presentation on 20 generators and 29 relators such that K/K' is elementary abelian of order 3^{20}. So K is infinite by Theorem 2.10, and hence G is infinite.

$(4,13\,|\,2,5)$: Let H be the subgroup generated by the elements

$$a,\quad b^2ab^{-1},\quad b^{-2}ab^3,\quad ba^{-1}b^2ab^{-1}ab^{-1},\quad b^{-1}ab^3ab^{-2}a^{-1}b,$$

$$b^{-5}ab^{-1}a^{-1}b,\quad b^{-1}ab^{-4}a^{-1}b^2a^{-1}b \quad\text{and}\quad b^{-1}ab^2a^{-1}b^2a^{-2}b^{-2}a^{-1}b.$$

Then H has index 26 in G, H/H' is isomorphic to $C_2 \times C_4^3$ and H'/H'' is isomorphic to $C_5^{67} \times C_\infty^{194}$. So G is infinite.

$(4,15\,|\,2,5)$: Let H be the subgroup generated by b, $a^2b^{-1}a$ and $a^2b^3a^{-1}$. Then H has index 6 in G and presentation

$$\langle x,y,z : x^{15} = y^2 = z^5 = (xyx^{-1}z^{-1})^2 = (xz^{-1}y)^3 = 1\rangle.$$

If we add the relation $z=1$, we get a presentation for the triangle group (2, 3, 15). So G is infinite.

$(4,17\,|\,2,5)$: This time we let H be the subgroup generated by a, $b^{-1}ab^{-3}$, $b^2a^{-1}b^3$ and $b^{-1}ab^2ab^{-3}a^{-1}b$. Then H has index 17 in G and has a presentation with 4 generators and 3 relators. So H is infinite, and hence G is infinite.

$(4, 19 \mid 2, 5)$: This time we take H to be the subgroup generated by the following elements:

$$a, \quad b^2ab^{-1}, \quad b^{-1}ab^4, \quad ba^{-1}b^2ab^{-1}ab^{-1}, \quad b^{-1}ab^{-2}ab^3a^{-1}b,$$

$$b^{-1}ab^{-1}ab^2a^{-1}b^2a^{-1}b^2, \quad b^{-1}ab^{-5}ab^{-1}a^{-1}ba^{-1}b,$$

$$b^{-1}aba^{-1}b^5ab^{-4}ab^{-1}a^{-1}b \quad \text{and} \quad b^{-2}ab^{-5}a^{-1}ba^{-1}b^3a^{-1}b^2.$$

We find that H has index 38 and H/H' is isomorphic to $C_2^2 \times C_4 \times C_5 \times C_\infty^2$, so that G is infinite. □

6. The groups $(4, m \mid n, k)$

In this section, we will consider the groups $(4, m \mid n, k)$ (with $m \geq 4$) not covered by Theorem 5.2. If $n = k = 2$, then the group is finite, and so we will assume that $k \geq n \geq 2$ and that $k \geq 3$. If m is even, or if k is even with $n = 2$, the group is covered by Corollary 2.5; this takes care of $(4, 6 \mid 2, k)$, $k \geq 3$, $(4, 4 \mid 3, k)$, $3 \leq k \leq 5$, $(4, 6 \mid 3, k)$, $3 \leq k \leq 5$, $(4, 10 \mid 2, 3)$, $(4, 12 \mid 2, 3)$, $(4, m \mid 2, 4)$, $m \geq 7$, $(4, 8 \mid 3, 4)$, $(4, 8 \mid 3, 5)$, $(4, 10 \mid 3, 5)$, $(4, 7 \mid 2, 6)$, $(4, 8 \mid 2, 5)$, $(4, 8 \mid 2, 6)$, $(4, 9 \mid 2, 6)$, $(4, 10 \mid 2, 5)$, $(4, 10 \mid 2, 6)$ and $(4, 11 \mid 2, 6)$, which are all infinite with the exception of $(4, 6 \mid 2, 3)$. On the other hand, Proposition 2.1 gives that $(4, 4 \mid 2, k)$, $k \geq 3$, and $(4, m \mid 2, 3)$, $7 \leq m \leq 9$, are finite. The group $(4, 5 \mid 2, k)$, $k \geq 3$, is finite if and only if $k \leq 5$ by Theorem 2.6.

This leaves the following groups, which we deal with on a case-by-case basis. In each case, we are considering the group G defined by the presentation

$$\langle a, b : a^4 = b^m = (ab)^n = (ab^{-1})^k = 1 \rangle.$$

$(4, 5 \mid 3, 3)$: The subgroup H generated by the elements a, b^2ab^{-1} and bab^3 has index 5 and presentation

$$\langle u, v, w : u^4 = v^2 = w^3 = (u^2v)^2 = (uw)^3 = (vw)^3 = (uvw^{-1})^3 = 1 \rangle.$$

If we let K be the subgroup of H generated by u, v, $w^{-1}uw$ and $w^{-1}vw$, then K has index 3 in H and presentation

$$\langle d, e, f, g : d^2 = e^2 = (de)^2 = f^4 = g^4 = (df^2)^2 = (eg^2)^2$$
$$= defdgef^{-1}g^{-1} = (fg)^4 = (degfgf)^2 = 1 \rangle.$$

Add the relations $f^2 = g^2 = (fg)^2 = 1$ to get a homomorphic image L with presentation

$$\langle d, e, f, g : d^2 = e^2 = (de)^2 = f^2 = g^2 = defdgefg = (fg)^2 = 1 \rangle.$$

The derived subgroup of L has index 16 in L and is free abelian of rank three, and so $(4, 5 \mid 3, 3)$ is infinite.

$(4,5\,|\,3,4)$: The subgroup H generated by the elements a, bab^{-1} and $b^{-1}aba^{-1}b^2$ has index 10 and presentation

$$\langle u,v,w : u^4 = w^4 = (uv^{-1})^4 = uv^{-1}wvwv^{-1}w^{-1}vw^{-1}u^{-1}v^{-1} = (uvu^{-1}v)^3 = 1\rangle.$$

If we add the relation $v = 1$ we see that G maps onto the free product $C_4 * C_4$, so that G is infinite.

$(4,5\,|\,3,5)$: We may define a homomorphism from G to the alternating group A_6 by $a \mapsto (1,\ 2,\ 5,\ 3)(4,\ 6)$ and $b \mapsto (1,\ 2,\ 3,\ 4,\ 5)$. This shows that a, b, ab and ab^{-1} really do have orders 4, 5, 3 and 5 respectively in G, so that G is infinite by Proposition 2.9.

$(4,11\,|\,2,3)$: If we add the extra relation

$$b^{-1}ab^3a^{-1}b^{-1}ab^3a^{-1}b^{-2}a^{-1}b^3ab^{-1}a^{-1}b^2a^{-1} = 1,$$

we get the group $PSL(2,23)$ of order 6072, and the kernel of the map from G onto $PSL(2,23)$ has infinite abelianization; so G is infinite.

$(4,m\,|\,3,3)$, $m \geq 7$: There is a natural homomorphism from $(4,m\,|\,3,3)$ onto $(2,m\,|\,3,3)$, i.e. onto the triangle group $(2,m,3)$, which is infinite for $m \geq 7$. So $(4,m\,|\,3,3)$ is infinite.

$(4,7\,|\,3,4)$: We may define a homomorphism from G to the alternating group A_8 by $a \mapsto (1\ 5\ 3\ 7)(2\ 8\ 4\ 6)$ and $b \mapsto (1\ 4\ 7\ 3\ 6\ 8\ 5)$. This shows that a, b, ab and ab^{-1} really do have orders 4, 7, 3 and 4, respectively, in G, so that G is infinite by Proposition 2.9.

$(4,9\,|\,3,4)$: We have a homomorphism from G onto A_9 defined by

$$a \mapsto (3,4,6,5)(8,9), \qquad b \mapsto (1,2,3,5,6,7,8,9,4).$$

So G does not collapse, and hence is infinite by Proposition 2.9.

$(4,m\,|\,3,4)$, $m \geq 10$: Since $(4,m\,|\,2,3)$ is infinite for $m \geq 10$ by Corollary 2.5 and Theorem 5.2, $(4,m\,|\,3,4)$ is infinite also.

$(4,7\,|\,3,5)$: We have a homomorphism from G onto A_7 defined by

$$a \mapsto (1,4,7,2)(3,6), \qquad b \mapsto (1,3,6,5,2,4,7).$$

So G does not collapse, and hence is infinite by Proposition 2.9.

$(4,9\,|\,3,5)$: We have a homomorphism from G to A_{27} defined by

$$a \mapsto (1,23,2,22)(3,13,7,9)(4,20,8,27)(5,19,16,26)(6,14,15,10)(11,24)(12,21)(17,18),$$
$$b \mapsto (1,10,15,2,12,27,25,26,23)(3,9,21,16,19,14,17,18,13)(4,7,6,11,24,22,5,8,20).$$

In fact, G is being mapped onto $U_4(2)$ here, where $U_4(2)$ is acting as a group of incidence-preserving permutations of the 27 lines of the generalized cubic surface in projective 3-space. In any case, G does not collapse, and hence is infinite by Proposition 2.9.

$(4,7\,|\,2,5)$: Let H be the subgroup generated by the elements

$$a, \quad b^2ab^{-1} \quad \text{and} \quad b^{-1}ab^2ab^{-3}a^{-1}b.$$

Then H has index 21 in G, H/H' is isomorphic to $C_2 \times C_2$, H'/H'' is isomorphic to $C_5 \times C_{10}$, and H'' has a presentation on 278 generators and 267 relators. So H'' is infinite, and hence G is infinite.

$(4,9\,|\,2,5)$: Let H be the subgroup generated by

$$a, \quad b^2a^2b^{-2}, \quad b^{-2}ab^3 \quad \text{and} \quad ba^{-1}b^2ab^{-2}.$$

Then H has index 15 in G and has presentation

$$\langle w,x,y,z : w^4 = x^2 = y^2 = (wz)^3 = (w^2zxz^{-1}xyz^{-1})^3$$
$$= w^2zxz^{-1}w^2zyxw^{-1}ywxyz^{-1} = 1\rangle.$$

Adding the relations $x = y = 1$ yields a presentation for the triangle group $(3,3,4)$, so that G is infinite.

$(4,11\,|\,2,5)$: Let H be the subgroup generated by

$$a, \quad b^2ab^{-1}, \quad b^{-2}ab^3, \quad b^{-1}ab^{-1}ab^2a^{-1}b, \quad b^{-1}ab^{-1}ab^2a^{-1}b,$$

$$b^{-1}aba^{-1}b^{-3}ab^{-1} \quad \text{and} \quad b^{-1}ab^3ab^{-3}a^{-1}b.$$

We find that H has index 22 in G, H/H' is isomorphic to $C_2 \times C_2 \times C_4$, and H'/H'' is isomorphic to $C_5^{17} \times C_\infty^{10}$. (In fact, H has a presentation on 28 generators and 19 relators.) So G is infinite.

7. The groups $(l, m\,|\,n, k)$ with $l > 4$

In this section, we will consider the groups $(l,m\,|\,n,k)$ (with $m \geq l > 4$) not covered by Theorem 5.2; in all such cases, we have that $n = 2$, and we will assume that this condition holds throughout this section. If (in addition) $k = 2$, then the group is finite, and so we will assume that $k \geq 3$. If any two of l,m and k are even, the result follows from Corollary 2.5; this takes care of $(6,6\,|\,2,3)$, $(6,8\,|\,2,3)$, $(6,10\,|\,2,3)$, $(6,12\,|\,2,3)$, $(6,m\,|\,2,4)$, $m \geq 6$, $(6,6\,|\,2,5)$, $(6,8\,|\,2,5)$, $(8,8\,|\,2,3)$, $(8,10\,|\,2,3)$, $(10,10\,|\,2,3)$, $(10,12\,|\,2,3)$ and $(10,m\,|\,2,4)$, $m \geq 10$, which are all infinite. The groups $(5,5\,|\,2,k)$, $k \geq 3$ are finite for $k = 3$ or $k = 4$ and infinite otherwise by Theorem 2.8; Theorem 2.8 also gives that the groups $(7,7\,|\,2,4)$, $m \geq 7$, $(9,9\,|\,2,3)$ and $(11,11\,|\,2,3)$ are infinite.

The groups $(5,m\,|\,2,3)$, $m > 5$, $(6,7\,|\,2,3)$ and $(7,7\,|\,2,3)$ are finite by Proposition 2.1, and $(7,8\,|\,2,3)$ is finite by Theorem 2.2. The group $(5,m\,|\,2,4)$, $m > 5$, is isomorphic to $(4,5\,|\,2,m)$ by Proposition 2.3, and so is infinite by Theorem 2.6. The group

$(5,m\,|\,2,5)$, $m>5$, is isomorphic to $(5,5\,|\,2,m)$ by Proposition 2.3, and so is infinite by Theorem 2.8. The group $(7,m\,|\,2,3)$, $m \geq 9$, is infinite by Theorem 2.7.

This leaves the following groups, which we deal with on a case-by-case basis. Again, we are considering the group G with presentation

$$\langle a,b : a^l = b^m = (ab)^n = (ab^{-1})^k = 1\rangle$$

in each case.

$(6,9\,|\,2,3)$: Let H be the subgroup generated by a^2b^{-1}, ba and b^3ab^{-2}. Then H has index 18 in G, H/H' is elementary abelian of order 8 and H'/H'' is free abelian of rank two, so that G is infinite.

$(6,11\,|\,2,3)$: Let H be the subgroup generated by ab^{-1}, a^2, $a^{-1}b^4a^{-2}b$ and $b^{-2}\,a^2b^{-1}a^{-1}b^2$. Then H has index 22 in G, H/H' is isomorphic to C_3, H'/H'' is isomorphic to C_2^6, and H''/H''' is infinite, so that G is infinite.

$(6,7\,|\,2,5)$: Let H be the subgroup generated by the elements a, b^2ab^{-1}, $b^{-1}ab^2$ and $ba^{-1}ba^2ba^{-1}b^{-1}ab^{-1}$, which has index 16 in G. Then the commutator subgroup H' of H has index 4 in H and infinite abelianization; so G is infinite.

$(8,9\,|\,2,3)$: Let H be the subgroup generated by a, $b^{-2}a^3b^{-2}$ and $b^3a^{-2}b^{-2}$, which has index 18 in G and presentation

$$\begin{aligned}\langle x,y,z : x^8 &= y^2 = z^8 = xzx^{-1}zyz^{-1}x^{-1}zxz^{-1}x^{-1}yz^{-2}\\ &= x^2z^{-1}x^{-1}zxz^{-1}x^{-1}z^2xzx^{-1}z^{-1}xz = 1\rangle.\end{aligned}$$

Adding the relation $x=z$ yields $\langle x,y : x^2 = y^2 = 1\rangle$, and hence G is infinite.

$(8,m\,|\,2,3)$, $m \geq 11$: Since $(4,m\,|\,2,3)$ is infinite for $m \geq 11$ by Corollary 2.5 and Theorem 5.2, these groups are infinite also.

$(9,10\,|\,2,3)$: If we add the relations

$$a^{-2}b^{-3}a^2b^{-1}a^{-1}bab^{-2}a^2b^{-1} = b^2a^{-3}b^{-2}a^{-1}b^2a^{-2}bab^{-1}a^{-1} = 1,$$

we get the group $PSL(2,19)$ of order 3420. The kernel of the homomorphism from G onto $PSL(2,19)$ has infinite abelianization, and so G is infinite.

$(10,11\,|\,2,3)$: Let H be the subgroup generated by a, b^2ab^{-1}, $b^{-1}a^2b^{-1}a^{-1}b$ and $b^{-2}a^2b^{-1}a^2b^{-1}a^{-1}ba^{-2}b^2$. Then H has index 22 in G and has a presentation on 4 generators and 9 relations. Using Quotpic, we find that H has 21 maps onto A_4. One of these maps has kernel with abelianization isomorphic to $C_{10}^2 \times C_\infty^3$, and hence G is infinite.

$(10,13\,|\,2,3)$: If H is the subgroup generated by a, b^2ab^{-1}, $b^{-2}ab^3$ and $b^{-3}a^3b^{-1}$, then H has index 13 in G and has a presentation on 4 generators and 8 relations. Using Quotpic, we find that H has several maps onto S_5. One of these maps has kernel with abelianization isomorphic to C_∞^{51}, and hence G is infinite.

Acknowledgements

The authors would like to thank Jim Howie for several helpful conversations in relation to the work discussed in this paper and the referee for several perceptive comments which improved the exposition; the second author would like to thank Hilary Craig for all her help and encouragement.

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